# Field theory of propagating reaction-diffusion fronts

C. Escudero
*Departamento de Física Fundamental, Universidad Nacional de Educación a Distancia, C/ Senda del Rey 9, 28040 Madrid, Spain*
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The problem of velocity selection of reaction-diffusion fronts has been widely investigated. While the mean-field limit results are well known theoretically, there is a lack of analytic progress in those cases in which fluctuations are to be taken into account. Here, we construct an analytic theory connecting the first principles of the reaction-diffusion process to an effective equation of motion via field-theoretic arguments, and we arrive at results already confirmed by numerical simulations.

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## I. INTRODUCTION

Reaction-diffusion front propagation in nonequilibrium systems is a topic that has been receiving increasing attention recently. The numerous possible applications of the theory, of systems like flames [1], bacterial colonies [2], or population genetics [3], is of course one of the reasons for this recent interest. One of the most common approaches to this problem has been the use of deterministic reaction-diffusion equations, like the Fisher equation [3]. This equation, which combines logistic growth with diffusion, is one of the most important mathematical models in biology and ecology [4]. In one spatial dimension, the Fisher equation reads

$$\partial_t U = D\partial_{xx} U + aU - bU^2. \tag{1}$$

One can think of this equation as a mean-field description of a reaction-diffusion process of a single species of random walkers $A$ undergoing the reactions of birth $A \to A+A$ at rate $a$ and annihilation $A+A \to \emptyset$ at rate $b/2$. The analysis of this equation is straightforward. Consider the boundary conditions $U \to b/a$ when $x \to -\infty$ and $U \to 0$ when $x \to \infty$. Thus the linearly stable phase $b/a$ invades the linearly unstable phase 0. Assuming a stationary front profile $U(x,t) = U(x - vt) = U(z)$ and shifting variables $x \to \sqrt{D/a}x$, $t \to t/a$, and $U \to (a/b)U$ we get

$$U'' + cU' + U - U^2 = 0, \tag{2}$$

where $c = v/\sqrt{Da}$. The velocity of the front is controlled by its edge; this means the region of the front that is closer to the unstable phase $U=0$. We can thus linearize Eq. (2) around this value to get

$$U'' + cU' + U = 0. \tag{3}$$

The only physically acceptable solution to this equation is

$$U(z) \sim e^{-\gamma z}, \tag{4}$$

and substituting Eq. (4) into Eq. (3) we get

$$c = \gamma + \frac{1}{\gamma} \tag{5}$$

for arbitrary $\gamma$. It is clear that the range of velocities is thus $c \geq 2$, and it was shown that the minimal velocity is selected in the long-time limit [5]. We can thus conclude that, in this limit, $v = 2\sqrt{Da}$.

We will now show that this picture changes strongly when internal fluctuations effects, due to the finitness and discretness of the reactants, are taken into account.

## II. FIELD THEORY

We will consider single-species particles $A$ undergoing reactions $A \to A+A$ at rate $\sigma$ and $A+A \to \emptyset$ at rate $\lambda$. Further, we suppose the particles $A$ performing a random walk in a one-dimensional lattice with lattice spacing $b$. The exact description of the problem is given by the following master equation:

$$\frac{dP(\{n_i\};t)}{dt} = \sum_i \left[ \left.\frac{dP(\{n_i\};t)}{dt}\right|_D + \left.\frac{dP(\{n_i\};t)}{dt}\right|_\sigma + \left.\frac{dP(\{n_i\};t)}{dt}\right|_\lambda \right], \tag{6}$$

with

$$\left.\frac{dP(\{n_i\};t)}{dt}\right|_D = \frac{D}{b^2}\sum_{\{e\}} [(n_e+1)P(\ldots,n_i-1,n_e+1,\ldots;t) - n_i P(\ldots,n_i,n_e,\ldots;t)], \tag{7}$$

where $\{e\}$ denotes the set of nearest-neighbor sites adjacent to $i$ and $D$ is the diffusion constant,

$$\left.\frac{dP(\{n_i\};t)}{dt}\right|_\sigma = \sigma[(n_i-1)P(\ldots,n_i-1,\ldots;t) - n_i P(\ldots,n_i,\ldots;t)], \tag{8}$$

and

$$\left.\frac{dP(\{n_i\};t)}{dt}\right|_\lambda = \lambda[(n_i+2)(n_i+1)P(\ldots,n_i+2,\ldots;t) - n_i(n_i-1)P(\ldots,n_i,\ldots;t)]. \tag{9}$$

For simplicity we will choose an uncorrelated Poisson distribution as initial condition for our master equation:

$$P(\{n_i\};t=0) = e^{-N(0)}\prod_i \frac{n_{0i}^{n_i}}{n_i!}, \tag{10}$$

where $N(0) = \Sigma_i n_{0i}$. We can map this master equation description of the system onto a quantum-field-theoretic problem.

 

This connection was first proposed by Doi [6], further elucidated by Peliti [7], and a deep generalization of it can be found in the influencing article by Cardy and Täuber [8]. We can write this theory in terms of the second-quantized bosonic operators

$$[a_i^\dagger, a_j] = \delta_{ij}, \quad [a_i, a_j] = 0, \quad [a_i^\dagger, a_j^\dagger] = 0, \quad a_i|0\rangle = 0, \tag{11}$$

whose effect is to create or to annihilate particles at the corresponding lattice site:

$$a_i^\dagger|\ldots, n_i, \ldots\rangle = |\ldots, n_i + 1, \ldots\rangle, \tag{12}$$

$$a_i|\ldots, n_i, \ldots\rangle = n_i|\ldots, n_i - 1, \ldots\rangle, \tag{13}$$

where we have defined the states as

$$|\{n_i\}\rangle = \prod_i (a_i^\dagger)^{n_i}|0\rangle. \tag{14}$$

Thus we can define the time-dependent state vector as

$$|\Phi(t)\rangle = \sum_{\{n_i\}} P(\{n_i\}; t)|\{n_i\}\rangle \tag{15}$$

and claim that it obeys the imaginary time Schrödinger equation

$$\frac{d}{dt}|\Phi(t)\rangle = -H|\Phi(t)\rangle, \tag{16}$$

with the Hamiltonian

$$H = \sum_i \left( -\frac{D}{b^2}\sum_{\{e\}} a_i^\dagger(a_e - a_i) - \lambda[1 - (a_i^\dagger)^2]a_i^2 + \sigma[1 - a_i^\dagger]a_i^\dagger a_i \right). \tag{17}$$

Note that we recover Eq. (6) if we substitute Eqs. (15) and (17) into Eq. (16). The time-dependent expectation value of an observable $O$ is given by

$$\langle O(t)\rangle = \sum_{\{n_i\}} O(\{n_i\})P(\{n_i\}; t). \tag{18}$$

To compute this quantity in the field-theoretic formalism we need to introduce the Glauber state:

$$\langle S| = \langle 0|\prod_i e^{a_i}, \quad \langle S|0\rangle = 0. \tag{19}$$

Note that this state is a left eigenstate of the creation operator with eigenvalue 1, implying that for any normal-ordered polynomial of the ladder operators one has

$$\langle S|Q(\{a_i^\dagger\}, \{a_i\}) = \langle S|Q(\{1\}, \{a_i\}). \tag{20}$$

Thus we can write expectation value Eq. (18) as

$$\langle O(t)\rangle = \langle S|O(\{a_i\})|\Phi(t)\rangle. \tag{21}$$

We can write this expectation value as coherent-state path integral:

$$\langle O(T)\rangle = \frac{\int \Pi_i D\hat{\psi}_i D\psi_i O(\{\psi_i\}) e^{-S[\hat{\psi}_i, \psi_i; T]}}{\int \Pi_i D\hat{\psi}_i D\psi_i e^{-S[\hat{\psi}_i, \psi_i; T]}}, \tag{22}$$

where the action is given by

$$S[\hat{\psi}_i, \psi_i; T] = \sum_i \left( \int_0^T dt \left[ \hat{\psi}_i(t)\frac{\partial}{\partial t}\psi_i(t) + H_i(\{\hat{\psi}_i(t)\}, \{\psi_i(t)\}) \right] \right). \tag{23}$$

Performing the continuum limit

$$\sum_i \to b^{-1}\int dx, \quad \psi_i(t) \to b\psi(x,t), \quad \hat{\psi}_i(t) \to \hat{\psi}(x,t),$$

$$\sum_{\{e\}} [\psi_e(t) - \psi_i(t)] \to b^3 \frac{\partial^2}{\partial x^2}\psi(x,t), \tag{24}$$

we get the action

$$S[\hat{\psi}, \psi; T] = \int dx \left[ \int_0^T dt \left( \hat{\psi}(x,t)\left[ \frac{\partial}{\partial t} - D\frac{\partial^2}{\partial x^2} \right]\psi(x,t) - \lambda_0[1 - \hat{\psi}(x,t)^2]\psi(x,t)^2 + \sigma[1 - \hat{\psi}(x,t)]\hat{\psi}(x,t)\psi(x,t) \right) \right], \tag{25}$$

where $\lambda_0 = b\lambda$.

## III. PERTURBATION THEORY

In order to study perturbatively this field theory we will perform a change of variables to rend the action, Eq. (25), dimensionless:

$$t \to \frac{t}{\sigma}, \quad x \to \sqrt{\frac{D}{\sigma}}x, \quad \hat{\psi} \to \hat{\psi}, \quad \psi \to \frac{\sigma}{\lambda}\psi. \tag{26}$$

This way we get

$$S(\hat{\psi}, \psi) = \epsilon^{-1}\int dxdt \left( \hat{\psi}(x,t)\left[ \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} \right]\psi(x,t) - [1 - \hat{\psi}(x,t)^2]\psi(x,t)^2 + [1 - \hat{\psi}(x,t)]\hat{\psi}(x,t)\psi(x,t) \right), \tag{27}$$

where $\epsilon^{-1} = \sqrt{D\sigma}/\lambda_0$. We will use from now on some standard results involving functionals and functional integrals in field theory; they can be seen, for instance, in [9]. The functional $Z(\hat{\eta}, \eta)$ with external sources is

$$Z(\hat{\eta}, \eta) = \int D\hat{\psi}(x,t)D\psi(x,t) \times \exp\left[ -\frac{1}{\epsilon}\left( S + \int dxdt[\hat{\eta}(x,t)\psi(x,t) + \hat{\psi}(x,t)\eta(x,t)] \right) \right]. \tag{28}$$

Using the steepest-descent procedure, we know that the func-

tional integral, Eq. (28), in the limit $\epsilon \to 0$ is dominated by the saddle points

$$\frac{\delta S}{\delta \psi(x,t)} = \hat{\eta}(x,t), \tag{29}$$

$$\frac{\delta S}{\delta \hat{\psi}(x,t)} = \eta(x,t). \tag{30}$$

Equations (29) and (30) in the absence of external sources ($\eta = \hat{\eta} = 0$) are the mean-field equations for the reaction-diffusion process. We will study perturbatively the functional integral, Eq. (28), in the neighborhood of the "classical field"—say, the solutions of the saddle-point equations (29) and (30): $\psi_c, \hat{\psi}_c$. Thus we will use the expansion

$$\psi = \psi_c + \sqrt{\epsilon}\chi, \tag{31}$$

$$\hat{\psi} = \hat{\psi}_c + \sqrt{\epsilon}\hat{\chi}, \tag{32}$$

and expanding the action in powers of $\epsilon$ we find

$$\begin{aligned} &S(\hat{\psi},\psi) - \eta\hat{\psi} - \hat{\eta}\psi \\ &\quad = S(\hat{\psi}_c,\psi_c) - \eta\hat{\psi}_c - \hat{\eta}\psi_c + \frac{\epsilon}{2}\int dx_1 dx_2 dt_1 dt_2 \\ &\qquad \times \Bigg[ \left.\frac{\delta^2 S}{\delta\psi(x_1,t_1)\delta\psi(x_2,t_2)}\right|_{\psi=\psi_c} \chi(x_1,t_1)\chi(x_2,t_2) \\ &\qquad + \left.\frac{\delta^2 S}{\delta\hat{\psi}(x_1,t_1)\delta\hat{\psi}(x_2,t_2)}\right|_{\hat{\psi}=\hat{\psi}_c} \hat{\chi}(x_1,t_1)\hat{\chi}(x_2,t_2) \\ &\qquad + 2\left.\frac{\delta^2 S}{\delta\hat{\psi}(x_1,t_1)\delta\psi(x_2,t_2)}\right|_{\psi=\psi_c,\hat{\psi}=\hat{\psi}_c} \hat{\chi}(x_1,t_1)\chi(x_2,t_2) \Bigg] \\ &\qquad + O(\epsilon^{3/2}). \end{aligned} \tag{33}$$

It is very important to note at this point that it is the edge of the front that leads to the marginal stability criterion—say, to the velocity selection. And the edge of the front is characterized by a low occupation number, so we can neglect terms proportional to $\chi^2$ and $\hat{\chi}^2$, which reflect the presence of more than one particle at the corresponding site, as we consider this event to be unlikely if we go far enough in the edge. We can see this clearly if we recall that $\chi$ and $\hat{\chi}$ are the eigenvalues of the annihilation and creation operators, respectively, and this way any of them squared reflects the possible presence of two particles in the same place.

The functional integral at this order becomes

$$\begin{aligned} Z(\hat{\eta},\eta) \sim Z_0(\hat{\eta},\eta) \int D\hat{\chi}D\chi \exp\Bigg( &- \int dx_1 dx_2 dt_1 dt_2 \\ &\times \frac{\delta^2 S}{\delta\hat{\psi}_c(x_1,t_1)\delta\psi_c(x_2,t_2)} \hat{\chi}(x_1,t_1)\chi(x_2,t_2) \Bigg), \end{aligned} \tag{34}$$

where

$$Z_0(\hat{\eta},\eta) = e^{(-1/\epsilon[S(\hat{\psi}_c,\psi_c) - \hat{\eta}\psi_c - \hat{\psi}_c\eta])}, \tag{35}$$

and therefore

$$Z(\hat{\eta},\eta) = NZ_0(\hat{\eta},\eta)\left[\det \frac{\delta^2 S}{\delta\hat{\psi}_c(x_1,t_1)\delta\psi_c(x_2,t_2)}\right]^{-1}. \tag{36}$$

The normalization factor $N$ is fixed by the condition $Z(0,0) = 1$.

The connected generating functional $W(\hat{\eta},\eta) = \epsilon \ln Z(\hat{\eta},\eta)$ at this order is then

$$W(\hat{\eta},\eta) = W_0(\hat{\eta},\eta) + \epsilon W_1(\hat{\eta},\eta) + O(\epsilon^2), \tag{37}$$

where

$$\begin{aligned} W_1(\hat{\eta},\eta) = -\Bigg[ &\left.\mathrm{tr}\ \ln \frac{\delta^2 S}{\delta\hat{\psi}_c(x_1,t_1)\delta\psi_c(x_2,t_2)}\right|_{\hat{\eta},\eta} \\ &- \left.\mathrm{tr}\ \ln \frac{\delta^2 S}{\delta\hat{\psi}_c(x_1,t_1)\delta\psi_c(x_2,t_2)}\right|_{\hat{\eta}=\eta=0} \Bigg]. \end{aligned} \tag{38}$$

Let us now perform the Legendre transformation

$$\begin{aligned} \Gamma(\hat{\phi},\phi) = \int dxdt[ &\eta(x,t)\hat{\phi}(x,t) + \hat{\eta}(x,t)\phi(x,t) - W_0(\hat{\eta},\eta) \\ &- \epsilon W_1(\hat{\eta},\eta)] + O(\epsilon^2). \end{aligned} \tag{39}$$

At one-loop order the one-particle-irreducible (1PI) functional is

$$\Gamma(\hat{\phi},\phi) = S(\hat{\phi},\phi) + \epsilon\Gamma_1(\hat{\phi},\phi) + O(\epsilon^2), \tag{40}$$

with

$$\Gamma_1(\hat{\phi},\phi) = \mathrm{tr}\left[\ln\frac{\delta^2 S}{\delta\hat{\phi}(x_1,t_1)\delta\phi(x_2,t_2)} - \ln \left.\frac{\delta^2 S}{\delta\hat{\phi}\delta\phi}\right|_{\hat{\phi}=\phi=0}\right]. \tag{41}$$

We can interpret the 1PI functional as an effective action that will lead us to new effective equations of motion:

$$\frac{\delta\Gamma}{\delta\phi} = 0, \tag{42}$$

$$\frac{\delta\Gamma}{\delta\hat{\phi}} = 0. \tag{43}$$

In our particular case, the action (27) gives

$$\frac{\delta^2 S}{\delta\hat{\phi}\delta\phi} = (\partial_t - \partial_{xx} + 1)\delta_2 + (4\hat{\phi}\phi - 2\hat{\phi})\delta_2, \tag{44}$$

where $\delta_2 = \delta(x_1 - x_2)\delta(t_1 - t_2)$. We conclude, therefore,

$$\begin{aligned} \Gamma_1(\hat{\phi},\phi) &= \int dxdt\langle x,t|\ln[1 + (\partial_t - \partial_{xx} + 1)^{-1}(4\hat{\phi}\phi - 2\hat{\phi})]|x,t\rangle \\ &= \int dxdt\langle x,t|(\partial_t - \partial_{xx} + 1)^{-1}(4\hat{\phi}\phi - 2\hat{\phi})|x,t\rangle, \end{aligned} \tag{45}$$

where we have made use of the low-occupation-number ap-

proximation for the front edge and a power series expansion of the logarithm. If we evaluate the matrix element, we get

$$\Gamma_1 = C \int dxdt(4\hat{\phi}\phi - 2\hat{\phi}), \tag{46}$$

where the constant $C$ is given by

$$\begin{aligned} C &= \frac{1}{(2\pi)^2}\int dp \int dw \frac{1}{-iw + p^2 + 1} \\ &= \frac{1}{(2\pi)^2}\int dp \int dw \int dt\Theta(t)e^{-(p^2+1)t}e^{iwt} \\ &= \frac{1}{(2\pi)^2}\int dp \int dt\Theta(t)e^{-(p^2+1)t}\int dwe^{iwt} \\ &= \frac{1}{(2\pi)^2}\int dp \int dt\Theta(t)e^{-(p^2+1)t}2\pi\delta(t) \\ &= \frac{1}{4\pi}\int dp \\ &= \frac{1}{2b_0}, \end{aligned} \tag{47}$$

where we have used the fact that $\Theta(0)=1/2$ (this property of the Heaviside $\Theta$ function can be found to be more rigorously proven in [9]) and that the integral over the whole momentum space is the volume of the first Brillouin zone, where $b_0$ is the dimensionless lattice spacing. Thus the effective action reads

$$\Gamma(\hat{\phi},\phi) = S(\hat{\phi},\phi) + \epsilon\frac{1}{2b_0}\int dxdt(4\hat{\phi}\phi - 2\hat{\phi}) + O(\epsilon^2), \tag{48}$$

and the new equations of motion (42) and (43) are

$$(\partial_t - \partial_{xx})\phi + 2\hat{\phi}\phi^2 - 2\hat{\phi}\phi + \phi + \epsilon\frac{1}{2b_0}(4\phi - 2) = 0, \tag{49}$$

$$-(\partial_t + \partial_{xx})\hat{\phi} - 2(1-\hat{\phi}^2)\phi + (1-\hat{\phi})\hat{\phi} + \epsilon\frac{1}{2b_0}(4\hat{\phi}) = 0. \tag{50}$$

If we recall that the field $\phi$ represents the expected value of the front density and that we are everywhere supposing that we are on the front edge, this quantity should be small enough to consider $\epsilon\phi$ neligible. This way we get that $\hat{\phi} = 1 + 2\epsilon/b_0$ solves Eq. (50). Substituting this result into Eq. (49) and taking into account that $b_0 = \sqrt{\sigma/D}b$ we get

$$\partial_t\phi = \partial_{xx}\phi + \phi - 2\phi^2 + \frac{\lambda}{\sigma}. \tag{51}$$

Note that in this case, contrary to the mean-field approach, a positive phase propagates into an (infinitesimaly) negative phase, something that is clearly unphysical. This is the deterministic expression of the compact support property of the front; i.e., the front becomes identically zero at a finite value of $x$. Actually, this property has been rigorously proven for this kind of fronts [10]. In this regime, since there are no particles, no reaction is possible, and only diffusion from adjacent sites is allowed, leading to the action

$$S_{diff} = \int dx \int dt[\hat{\phi}(\partial_t - \partial_{xx})\phi]. \tag{52}$$

Since this action is quadratic, it is the effective action to any order, and this implies that the effective equation of motion for the front propagation at first order in $\lambda/\sigma$ is

$$\partial_t\phi = \partial_{xx}\phi + \left(\phi - 2\phi^2 + \frac{\lambda}{\sigma}\right)\Theta(\phi). \tag{53}$$

This last derivation deserves a further explanation. It may be surprising to the reader that Eq. (51) performs such an unphysical behavior, but it is actually what one would expect *a priori*. Indeed, the analysis of an effective action commonly yields a shift of the fixed points of the original one; that is what has happened here. This suggests that the correct physical interpretation of the problem should have been that corresponding to a moving boundary one. That is, at the beginning, we should have taken into account two different actions, one for the space full of particles and one for the empty space, and study the propagation of the boundary between them. This preserves the physical meaning all along the derivation. It might be desirable to solve this problem without splitting it into two parts, something that maybe could be done by using stochastic differential equations [12,13].

## IV. FRONT PROPAGATION AND VELOCITY SELECTION

To study how the front propagates let us perform the change of variables $\phi = u - \lambda/\sigma$. At leading order, Eq. (53) becomes

$$\partial_t u = \partial_{xx}u + (u - 2u^2)\Theta\left(u - \frac{\lambda}{\sigma}\right). \tag{54}$$

Clearly, the fields $u$ and $\phi$ propagate at the same speed. Equation (54) was heuristically proposed and studied by Brunet and Derrida [11], and we will summarize the main conclusions of their work. We will consider that for sufficiently long times the front will converge to a stationary shape—that is, $u(x,t)=u(x-ct)=u(z)$—and Eq. (54) becomes

$$u'' + cu' + (u - 2u^2)\Theta\left(u - \frac{\lambda}{\sigma}\right) = 0, \tag{55}$$

where $c$ is the dimensionless front speed. We can distinguish between three regions in the front edge: in the first one, the equation of motion is given by

$$u'' + cu' + u - 2u^2 = 0; \tag{56}$$

in the second one, the field is small enough that we can linearize to get

$$u'' + cu' + u = 0; \tag{57}$$

and in the third one $u < \lambda/\sigma$, so we get

$$u'' + cu' = 0. \tag{58}$$

We impose as boundary conditions the continuity of the first derivative between the boundaries of the three different regions. It can be shown [11] that this leads to a dimensionless velocity

$$c = 2 - \frac{\pi^2}{\ln^2(\lambda/\sigma)} \tag{59}$$

at first order in $\lambda/\sigma$. The front velocity with the corresponding dimensions is thus

$$v = \sqrt{D\sigma}\left(2 - \frac{\pi^2}{\ln^2(\lambda/\sigma)}\right). \tag{60}$$

This correction has already been confirmed in numerical simulations [11–13], and it shows a very slow convergence to the mean-field velocity in the limit $\lambda/\sigma \to 0$. This shows that the discreteness of the reaction process strongly shifts the velocity to a slower one.

As a final remark, we would like to underline that the cutoff derived is not a particularity of the low-dimensional topology of the problem. Indeed, if one considers the $d$-dimensional problem and performs all the calculations shown here for the particular case $d=1$, one arrives at the dimensionless equation

$$\partial_t \phi = \nabla^2 \phi + \left(\phi - 2\phi^2 + \frac{\lambda}{\sigma}\right)\Theta(\phi), \tag{61}$$

locally describing the edge of the front. The reason for this independence between the appearance of the cutoff in the reaction term and the dimensionality of the system is due to the physical origin of the cutoff. It appears as a consequence of the physical fact that far enough in the right spatial direction (the direction of propagation of the front) there must be no particles. This is, of course, totally independent of the spatial dimension of the system. However, the effect of the cutoff on the dynamics of the front does strongly depend on the dimensionality. In one dimension, we have observed a strong shift in the velocity of the front, while in two dimensions the effect is even stronger and the presence of the cutoff is the only cause responsible for the formation of diffusive instabilities [14]. It would be very interesting to analyze the effects of the cutoff in dimensions above $d=2$, to see if new phenomenology develops or, on the contrary, there is a return to the mean field.

## V. CONCLUSIONS

In this work, we have derived from first principles a field theory of reaction-diffusion particles, and we have used it to study reaction-diffusion propagating fronts. This kind of fronts has been traditionally studied using deterministic reaction-diffusion equations, like the Fisher equation, which consider an infinite number of particles. We performed a perturbation expansion in the ratio between the annihilation and birth rates that separates the mean-field regime from the real discrete process and studied the first-order corrections to the equation of motion. A cutoff in the reaction term, a mechanism that has already been heuristically proposed, appeared in a natural way within our formalism and led us to the velocity corrections already found in numerical simulations.

Our first-principles analytically derived theory also allowed us to understand the fundamental reasons that lead to the velocity shift. It is the compact support property of the front—i.e., the fact that the field is identically zero far enough in the spatial axis—that causes such a dramatic effect in a pulled front like ours.

It is also interesting to compare this work to a former one in the same direction [15] which tries to derive a cutoff in the reaction term, albeit for a different system. The main difference between both works is, from our point of view, that while this article concerns a system in continuum space, the other deals with a lattice. This difference becomes fundamental since the calculations were performed using Stratonovich stochastic calculus, valid in the lattice, but ill posed in a continuum space [16].

Many questions are still to be answered. Different reaction shemes are to be explored; also, the opposite limit (the annihilation rate large compared to the birth rate) is only conjectured [13], but not analytically found. Of course, the higher dimensionality of the front is a very interesting problem, where new phenomenology does appear, as shown by Kessler and Levine [14]. We hope that this and former works will encourage the reader to attempt to solve these and different problems that appear in this subject.

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